High Probability Complexity Bounds for Trust-Region Methods with Noisy Oracles

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Collaborators

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Liyuan Cao, Albert S. Berahas, and Katya Scheinberg. First-and second-order high probability complexity bounds for trust-region methods with noisy oracles. *arXiv preprint arXiv:2205.03667*, 2022.

The Problem

$$\min_{x \in \mathbb{R}^n} \phi(x)$$

ϕ follows common assumptions

$$\phi(x) \geq \hat{\phi} \text{ for all } x \in \mathbb{R}^n,$$
$$\|\nabla\phi(x) - \nabla\phi(y)\| \leq L_1\|x - y\| \text{ for all } (x, y) \in \mathbb{R}^n \times \mathbb{R}^n,$$

but we only have access to

$$\begin{cases} f_k \\ g_k \\ H_k \end{cases} \text{instead of} \begin{cases} \phi(x_k) \\ \nabla\phi(x_k) \\ \nabla^2\phi(x_k). \end{cases}$$

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A line of work:

- **algorithm** TR method modified to handle noise.
- **noise** Weaker assumptions in more recent work.
- **result** Stronger results in more recent work.

The Trust-Region Method

Algorithm: Modified First-Order Trust-Region Method

Inputs: Starting point x_0, initial trust region radius δ_0, tolerance parameter r, and hyperparameters $\eta_1 > 0, \eta_2 > 0$, $\gamma \in (0,1)$ for controlling the trust region radius.

for $k = 0, 1, 2, \ldots$ **do**

Build a quadratic model $m_k(x_k + s) = \phi(x_k) + \langle g_k, s\rangle + 0.5\,\langle H_k s, s\rangle$

Compute s_k by approximately minimizing m_k in $B(x_k, \delta_k)$ so that it satisfies the *Cauchy decrease condition*

$$m_k(x_k) - m_k(x_k + s_k) \geq \frac{1}{2}\|g_k\| \min\left\{\frac{\|g_k\|}{\|H_k\|}, \delta_k\right\}.$$

Compute

$$\rho_k = \frac{f_k - f_k^+ + r}{m_k(x_k) - m_k(x_k + s_k)}$$

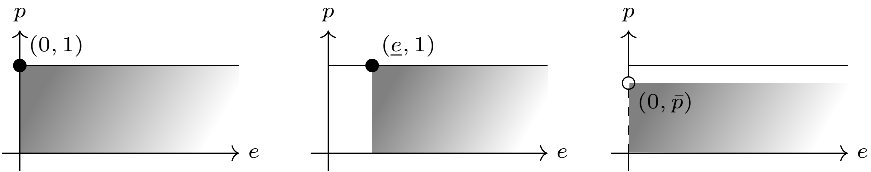
and update x and δ

$$(x_{k+1}, \delta_{k+1}) = \begin{cases} (x_k + s_k, \gamma^{-1}\delta_k) & \text{if } \rho_k \geq \eta_1 \text{ and } \|g_k\| \geq \eta_2\delta_k \\ (x_k + s_k, \gamma\delta_k) & \text{if } \rho_k \geq \eta_1 \text{ and } \|g_k\| < \eta_2\delta_k \\ (x_k, \gamma\delta_k) & \text{if } \rho_k < \eta_1. \end{cases}$$

Stochastic Oracle

Let $\varphi^{(j)}\left(x_k, \xi_k^{(j)}, \mathcal{S}_k^{(j)}\right)$ be the jth-order oracle that returns an estimate of $\nabla^j \phi(x_k)$ such that for all $(e, p) \in \mathcal{S}_k^{(j)} \subseteq [0, \infty) \times [0, 1]$,

$$\mathbb{P}_{\xi_k^{(j)}}\left\{\left\|\varphi^{(j)}\left(x_k, \xi_k^{(j)}, \mathcal{S}^{(j)}\right) - \nabla^j \phi(x_k)\right\| \leq e \,\middle|\, \mathcal{F}_k\right\} \geq p$$



(a) $\mathcal{S}_k^{(j)} \ni (0, 1)$
exact

(b) $\mathcal{S}_k^{(j)} = [\underline{e}, +\infty) \times [0, 1]$
bounded

(c) $\mathcal{S}_k^{(j)} = (0, +\infty) \times [0, \bar{p}]$
probabilistically sufficiently accurate

The Goal of the Analysis

ⓒ convergence

$$\liminf_{k\to\infty} \|\nabla\phi(x_k)\| \le \epsilon$$

ⓔ expected complexity

$$\mathbb{E}\min\{k : \|\nabla\phi(X_k)\| \le \epsilon\} = \mathcal{O}(1/\epsilon^2)$$

ⓗ high probability convergence

$$\mathbb{P}\left\{\min\{\|\nabla\phi(X_k)\| :\ 0 \le k \le T-1\} < \epsilon\right\}$$
$$\ge \text{a function of } T \text{ the converges to 1 as } T \text{ increase}$$

for some sufficiently large ϵ.

$\mathcal{S}^{(0)} = [0, \infty) \times [0, 1]$ and $\mathcal{S}^{(1)} = (0, \infty) \times [0, \bar{p}_1]$ for sufficiently large $\bar{p}_1$:

- ⓒ Afonso S Bandeira, Katya Scheinberg, and Luis Nunes Vicente. Convergence of trust-region methods based on probabilistic models. *SIAM Journal on Optimization*, 24(3):1238–1264, 2014.
- ⓗ Serge Gratton, Clement W Royer, Luis N Vicente, and Zaikun Zhang. Complexity and global rates of trust-region methods base on probabilistic models. *IMA Journal of Numerical Analysis*, 38(3):1579-1597, 2018.

$\mathcal{S}^{(j)} = (0, \infty) \times [0, \bar{p}_j]$ for sufficiently large $\bar{p}_j$, $j = 0, 1$:

- ⓒ Ruobing Chen, Matt Menickelly, and Katya Scheinberg. Stochastic optimization using a trust-region method and random models. *Mathematical Programming*, 169(2):447–487, 2018.

$\mathcal{S}^{(j)} = (0, \infty) \times [0, \bar{p}_j]$ for sufficiently large $\bar{p}_j$, $j = 0, 1, 2$ and $\mathbb{E}_{\xi_0}|f_k - \phi(x_k)| \leq C_0$:

- ⓔ Jose Blanchet, Coralia Cartis, Matt Menickelly, and Katya Scheinberg. Convergence rate analysis of a stochastic trust-region method via supermartingales. *INFORMS Journal on Optimization*, 1(2):92–119, 2019.

$\mathcal{S}^{(0)} = [\epsilon_f, \infty) \times [0, 1]$ and $\mathcal{S}^{(1)} = [\epsilon_g, \infty) \times [0, 1]$:

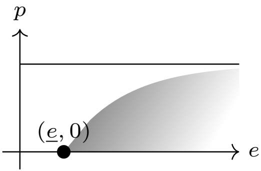
- ⓔ Shigeng Sun and Jorge Nocedal. A trust region method for the optimization of noisy functions. *arXiv preprint arXiv:2201.00973*, 2022.

We assume for all k:

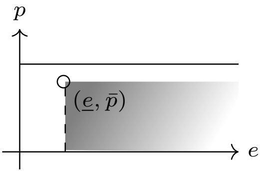
$$\left.\begin{array}{l}\mathbb{P}\left\{|f_k-\phi(x_k)|\le e\right\}\\ \mathbb{P}\left\{|f_k^+-\phi(x_k+s_k)|\le e\right\}\end{array}\right\}\ge \exp(a(\epsilon_f-e)) \qquad \text{unbounded noise,}$$

$$\mathbb{P}\left\{\|g_k-\nabla\phi(x_k)\|\le \kappa_{\text{eg}}\delta_k+\epsilon_g\right\}\ge p_1 \qquad \text{irreducible noise,}$$

$\|H_k\|\le\kappa_{\text{bhm}}$ for some constant κ_{bhm} (bound on hessian of model).



(a) $\mathcal{S}_k^{(0)}=\{(e,p):e\ge\underline{e}=\epsilon_f, p\le 1-\exp(a(\underline{e}-e))\}$



(b) $\mathcal{S}_k^{(1)}=(\underline{e},+\infty)\times[0,\bar{p}]$ with $\underline{e}=\kappa_{\text{eg}}\delta_k+\epsilon_g$ and $\bar{p}=p_1$

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for $k = 0, 1, 2, \dots$ **do**

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Compute s_k by approximately minimizing m_k in $B(x_k, \delta_k)$ so that it satisfies the *Cauchy decrease condition*

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Compute

$$\rho_k = \frac{f_k - f_k^+ + r}{m_k(x_k) - m_k(x_k + s_k)}$$

and update x and δ

$$(x_{k+1}, \delta_{k+1}) = \begin{cases} (x_k + s_k, \gamma^{-1}\delta_k) & \text{if } \rho_k \geq \eta_1 \text{ and } \|g_k\| \geq \eta_2 \delta_k \\ (x_k + s_k, \gamma \delta_k) & \text{if } \rho_k \geq \eta_1 \text{ and } \|g_k\| < \eta_2 \delta_k \\ (x_k, \gamma \delta_k) & \text{if } \rho_k < \eta_1. \end{cases}$$

random variables:	X_k	X_k^+	$\mathcal{E}_k$	$\mathcal{E}_k^+$	
realizations:	x_k	$x_k + s_k$	$\lvert f_k - \phi(x_k)\rvert$	$\lvert f_k^+ - \phi(x_k + s_k)\rvert$	
random variables:	M_k	∇M_k	$\nabla^2 M_k$	Δ_k	ρ_k
realizations:	m_k	g_k	H_k	δ_k	ρ_k

Define

$$I_k = \mathbb{1}\{\|\nabla M_k - \nabla\phi(X_k)\| \le \kappa_{\text{eg}}\Delta_k + \epsilon_g\} \qquad \text{gradient sufficiently accurate}$$

$$J_k = \mathbb{1}\{\mathcal{E}_k + \mathcal{E}_k^+ \le r\} \qquad \text{zeroth-order noise compensated}$$

$$\Lambda_k = \mathbb{1}\{\Delta_k > \bar{\Delta}\} \qquad \text{large TR radius}$$

$$\Theta_k = \mathbb{1}\{\rho_k \ge \eta_1 \text{ and } \|\nabla M_k\| \ge \eta_2\Delta_k\} \qquad \text{successful step}$$

$$\Theta_k' = \mathbb{1}\{\rho_k \ge \eta_1\} \qquad \text{accepted step}$$

where $\bar{\Delta} = C_1 \min_{0\le k\le T-1} \|\nabla\phi(X_k)\| - C_2\epsilon_g$.

Classification of Iterations

<table>
<tr><th></th><th colspan="3">$I_k = 1, J_k = 1$</th><th colspan="3">$I_k = 1, J_k = 0$</th><th colspan="3">$I_k = 0, J_k = 1$</th><th colspan="3">$I_k = 0, J_k = 0$</th></tr>
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Lemma (sufficient condition for successful step)

If $I_k J_k = 1$ *and* $\Lambda_k = 0$ *then* $\Theta_k = 1$.

Lemma (progress made in each iteration)

Let $h(\delta) = C_3\delta^2$. *Then we have*

$$\phi(X_k) - \phi(X_{k+1}) \geq \begin{cases} h(\Delta_k) - \mathcal{E}_k - \mathcal{E}_k^+ - r & \text{if } \Theta_k = 1 \text{ (successful)} \\ -\mathcal{E}_k - \mathcal{E}_k^+ - r & \text{if } \Theta'_k = 1 \text{ (accepted)} \\ 0 & \text{if } \Theta'_k = 0 \text{ (rejected).} \end{cases}$$

Lemma (total progress)

$$h(\gamma\bar{\Delta}) \sum_{k=0}^{T-1} \Theta_k \Lambda_k \leq \phi(x_0) - \hat{\phi} + \sum_{k=0}^{T-1} \Theta'_k \left(\mathcal{E}_k + \mathcal{E}_k^+ + r\right).$$

Lemma (total loss)

For any $t \geq 0$,

$$\mathbb{P}\left\{\sum_{k=0}^{T-1}\left(\mathcal{E}_k + \mathcal{E}_k^+ + r\right) \geq T(4/a + 2\epsilon_f + r) + t\right\} \leq \exp\left(-\frac{a}{4}t\right).$$

Let $t = rT$.

Lemma (total progress)

$$\begin{aligned} h(\gamma\bar{\Delta})\sum_{k=0}^{T-1}\Theta_k\Lambda_k &\leq \phi(x_0) - \hat{\phi} + \sum_{k=0}^{T-1}\Theta'_k\left(\mathcal{E}_k + \mathcal{E}_k^+ + r\right) \\ &< \phi(x_0) - \hat{\phi} + T(4/a + 2\epsilon_f + 2r) \end{aligned}$$

with probability at least $1 - \exp\left(-\frac{ar}{4}T\right)$.

$$h(\gamma\bar{\Delta}) = \gamma^2 C_3\left(C_1 \min_{0\leq k\leq T-1}\|\nabla\phi(X_k)\| - C_2\epsilon_g\right)^2$$

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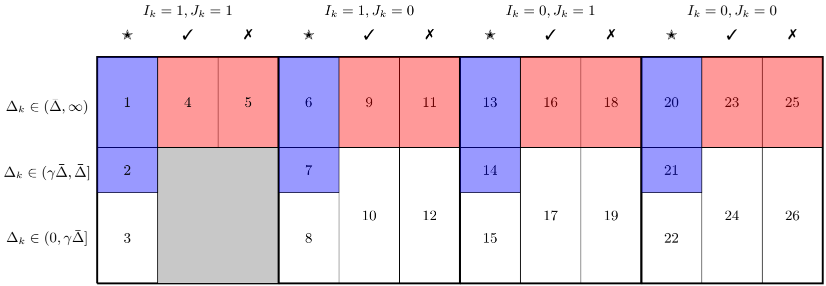
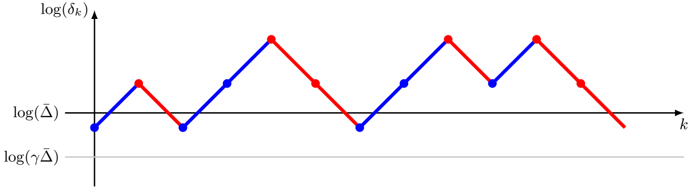
Lemma (**total progress**)

$$h(\gamma\bar{\Delta}) \sum_{k=0}^{T-1} \Theta_k \Lambda_k \leq \phi(x_0) - \hat{\phi} + T(4/a + 2\epsilon_f + 2r).$$

Ups and Downs of the Radius





$$\sum_{k=0}^{T-1} (1-\Theta_k)\Lambda_k < \sum_{k=0}^{T-1} \Theta_k \Lambda_k + \min\left\{\log_\gamma\left(\frac{\delta_0}{\bar{\Delta}}\right), 0\right\} + 1$$

Downs and Ups of the Radius

	$I_k=1, J_k=1$			$I_k=1, J_k=0$			$I_k=0, J_k=1$			$I_k=0, J_k=0$		
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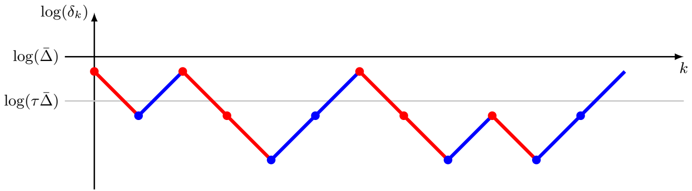

$$\sum_{k=0}^{T-1} \Theta_k (1-\Lambda_k) < \sum_{k=0}^{T-1} (1-\Theta_k)(1-\Lambda_k) + \min\left\{ \log_\gamma \left(\frac{\bar{\Delta}}{\delta_0} \right), 0 \right\} + 1$$

Iterations with Sufficiently Accurate Gradient Estimate

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Lemma

Assume $\mathbb{P}\{I_k = 1 \mid \mathcal{F}_k\} \geq p_1$ *holds. By Azuma-Hoeffding inequality, for any positive integer* T *and any* $\hat{p}_1 \in [0, p_1]$ *we have*

$$\mathbb{P}\left\{\sum_{k=0}^{T-1} I_k > \hat{p}_1 T\right\} \geq 1 - \exp\left(-\frac{(1 - \hat{p}_1/p_1)^2}{2} T\right).$$

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Lemma

Assume both $\mathbb{P}\{\mathcal{E}_k > t\}$ *and* $\mathbb{P}\{\mathcal{E}_k^+ > t\}$ *are* $\leq \exp(a(\epsilon_f - t))$*. Let* $p_0 = 1 - 2\exp(a[\epsilon_f - r/2])$*. For any positive integer* T *and any* $\hat{p}_0 \in [0, p_0]$*, we have*

$$\mathbb{P}\left\{\sum_{k=0}^{T-1} J_k > \hat{p}_0 T\right\} \geq 1 - \exp\left(-\frac{(1-\hat{p}_0/p_0)^2}{2}T\right).$$

Analysis

$$\sum_{k=0}^{T-1}(1-\Theta_k)\Lambda_k < \sum_{k=0}^{T-1}\Theta_k\Lambda_k + \min\left\{\log_\gamma\left(\frac{\delta_0}{\bar{\Delta}}\right), 0\right\} + 1$$

$$\sum_{k=0}^{T-1}\Theta_k(1-\Lambda_k) < \sum_{k=0}^{T-1}(1-\Theta_k)(1-\Lambda_k) + \min\left\{\log_\gamma\left(\frac{\bar{\Delta}}{\delta_0}\right), 0\right\} + 1$$

$$\mathbb{P}\left\{\sum_{k=0}^{T-1} I_k > \hat{p}_1 T\right\} \geq 1 - \exp\left(-\frac{(1-\hat{p}_1/p_1)^2}{2}T\right)$$

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$$\Downarrow$$

$$\mathbb{P}\left\{\sum_{k=0}^{T-1}\Theta_k\Lambda_k > \left(\hat{p}_0 + \hat{p}_1 - \frac{3}{2}\right)T - \frac{1}{2}\left|\log_\gamma\frac{\bar{\Delta}}{\delta_0}\right| - \frac{1}{2}\right\}$$

$$\geq 1 - \exp\left(-\frac{(1-\hat{p}_1/p_1)^2}{2}T\right) - \exp\left(-\frac{(1-\hat{p}_0/p_0)^2}{2}T\right)$$

Main Result

Let $t = rT$.

Theorem

Let assumptions hold. Given any $\epsilon > \sqrt{\frac{4\epsilon_f + 8/a + 2r}{C_3\gamma^2 C_1^2(2p_0 + 2p_1 - 3)}} + \frac{C_2}{C_1}\epsilon_g$*, we have*

$$\mathbb{P}\left\{\min\{\|\nabla\phi(X_k)\| : \ 0 \le k \le T-1\} \le \epsilon\right\} \ge$$
$$1 - \exp\left(-\frac{(1-\hat{p}_1/p_1)^2}{2}T\right) - \exp\left(-\frac{(1-\hat{p}_0/p_0)^2}{2}T\right) - \exp\left(-\frac{ar}{4}T\right)$$

for any $\hat{p}_0$ *and* $\hat{p}_1$ *such that* $\hat{p}_0 + \hat{p}_1 \in \left(\frac{3}{2} + \frac{2\epsilon_f + 4/a + r}{C_3\gamma^2(C_1\epsilon - C_2\epsilon_g)^2}, p_0 + p_1\right]$*, any* $t \ge 0$*, and any*

$$T \ge \left(\hat{p}_0 + \hat{p}_1 - \frac{3}{2} - \frac{2\epsilon_f + 4/a + 2r}{C_3\gamma^2(C_1\epsilon - C_2\epsilon_g)^2}\right)^{-1}$$
$$\left[\frac{\phi(x_0) - \hat{\phi}}{C_3\gamma^2(C_1\epsilon - C_2\epsilon_g)^2} + \frac{1}{2}\left|\log_\gamma \frac{C_1\epsilon - C_2\epsilon_g}{\delta_0}\right| + \frac{1}{2}\right] = \bar{\mathcal{O}}(\epsilon^{-2}).$$

Other Results

- Analyses under bounded noise assumption.
- Second-order TR method and analysis.
- Numerically testing the strength of the theoretical results.
- Experimenting with different values for r.